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STRUCTURAL DESIGN BY SYSTEMATIC SYNTHESIS

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SYNOPSIS

The use of analysis as a tool in structural design is well known. However, the really effective use of methods of analysis requires that rational methods of directed redesign be developed. Systematic structural synthesis may be defined as the rational, directed, evolution of a structural configuration which, in terms of a defined criterion, efficiently performs a set of specified functional purposes. The structural design problem is viewed as a problem in the programming of interdependent activities involving: requirements and specifications, a technology governing the behavior of the system, and a criterion for evaluating the relative merit of alternate designs.

It is assumed that a structure is to be designed to perform satisfactorily under several distinct design loading conditions. Limitations on stress and displacement which may be different for each element for each load condition are selected. A method for systematically converging on an optimum design in the sense of minimum total structural weight is described. Results obtained for elementary but illustrative examples using an IBM 653 digital computer are given. The emphasis throughout is on clearly defining the redesigned process in order to make possible automation of the design cycle rather than just the analysis phase.

INTRODUCTION

Methods of analysis which adequately predict the behavior of many structural systems are well known. The huge strides that have been made in the digital computing field have led to the routine use of reliable methods previously considered impractical because of the computational effort involved. However, the really effective use of structural analysis requires that rational methods of directed redesign be

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developed. There has been a tendency to regard a problem as solved when a reliable method of analysis has been developed, while in fact the availability of a reliable method of analysis is only a prerequisite to tackling the task of design synthesis.

The purpose of structural design is to evolve structural configurations which efficiently perform a set of specified functional purposes. Two common characteristics which often complicate the design problem are:

a. design requirements calling for structural integrity in a large number of distinct loading conditions, and
b. the use of highly indeterminate configurations (multiple load path structural systems).

The structural design cycle can be thought of in terms of three main phases:

1. Establish a trial design.
2. Carry out an analysis based on this trial design, and
3. Based on the analysis, modify the trial design as required.

With the benefit of experience and judgment, it is usually possible to establish a conservative trial design.

In the past several years much progress has been made in the direction of systematizing the analysis of highly indeterminate structures. Coupled with the use of large scale digital computing facilities, these methods have proven to be an effective analytical tool. The time scale for a trip around the design cycle has been effectively compressed thus permitting analysis to help in obtaining not only adequacy but efficiency. By and large these improvements in the design cycle process have been limited to improving, systematizing, and speeding up the analysis of a trial design using automatic computing facilities. After each analysis the redesign process takes place. The redesign process in general is not clearly defined, but rather it is an artful combination of judgment, experience, and often courage. The problem of stating mathematically the philosophy or basis on which redesign decisions are made has been an obstacle to the development of methods of structural synthesis. The problem is twofold. First an appropriate design philosophy must be adopted, and second a means of mathematically stating the philosophy in terms of a criterion by means of which choices can be made must be established.

STATEMENT OF THE GENERAL PROBLEM

Systematic structural synthesis may be defined as the rational directed evolution of a structural configuration which, in terms of a defined criterion, efficiently performs a set of specified functional purposes. In economics much work has been done in an area referred to

as the programming of interdependent activities.[2] (1,2,3,4, and 5) The structural design problem can be looked at as a problem in the programming of interdependent activities. Three types of considerations are involved in such problems: a specified set of requirements, a given technology, and a criterion by means of which choices can be made between various solutions. The analogous phases in the structural design problem may be postulated as follows:

I. Specifications and Requirements.—

A. The design load system is made up of several distinct design load conditions. Each load condition may involve several mechanical loads as well as a structural temperature distribution. The optimum design will be a balanced design for the entire design load system made up of several conditions. Note that the minimum weight optimum design of a statically determinate structure has the property that each member is fully utilized in at least one load condition. While this is a valid criterion for statically determinate structures, it is not in general valid for statically indeterminate structures. In this connection, it should be recognized that if an optimum design is sought using each design load condition separately, several distinct incompatible designs will result.

B. It is required that the stresses and deflections do not exceed certain prescribed values. In the case of an elastic stability constraint, the allowable stress will depend on the design parameters. An instability constraint can be thought of as a displacement constraint expressed in terms of an allowable stress. Limitations on both stresses and displacements are prescribed and may differ in each element for each loading condition. For instance, the allowable stresses in an element of the structure might be substantially different due to different temperature conditions. Also, allowable stresses could be set at lower levels for load conditions which occur frequently while higher allowables are used for load conditions of infrequent occurrence. Thus weight penalties attending fatigue and elevated temperature conditions could be held to a minimum. In general, the limitations placed on stresses and displacements differ for each load condition for each element of the structure and the method evolved should take this into account.

C. It is required that the size of certain elements in the structure be greater than a specified minimum, or less than a specified maximum. It is possible that an optimizing process will cause certain elements to vanish or become very large. Frequently this cannot be permitted, and it is necessary to impose constraints on the design process in the form of minimum and maximum element sizes.

II. Technology.—

The appropriate method of analysis for the structural system considered is a component part of the structural synthesis. Different methods of analysis will be appropriate depending upon the class of

2. While linear programming is the more common name given this area of endeavor, it is avoided herein because of the unnecessary restrictions implied by the word linear.

structure involved and the design philosophy employed. In many cases the highly developed methods of lumped elastic analysis will be suitable. This paper is restricted to a consideration of structural systems which can be analyzed employing the methods of lumped elastic analysis.

In the last decade the use of lumped element structural idealizations followed by matrix formulation of the structural analysis problem has been highly developed and extensively used in airframe design. (6,7,8, 9,10,11 to mention only a few.) All of these methods have at least two things in common. First, the state of stress within any one lumped element is assumed to be fully described by one unknown and second, the displacement pattern of the structure is assumed to be described by the displacement of a finite number of discrete points. On the other hand, if a design philosophy based on collapse at ultimate load is to be used then the methods of limit analysis should be used. Pearson (12) has given a method of implementing the structural synthesis concept which is based on the limit design philosophy.

III. Criterion.—

In many important structural design areas the minimization of weight is important. It should be noted that a minimum weight basis for evaluating merit is probably the most readily stated and it is certainly of great importance in the design of flight vehicles. Of course, it is theoretically possible to employ criteria other than minimum weight. If, for example, enough is known about the factors influencing cost, it would be possible to systematically seek a design which would minimize cost. Even when using minimum weight as a basis of choice the degree to which design parameters are prespecified in the configuration can be used as a means of building in economy through simplification and standardization.

The general problem can now be stated mathematically. The formulation will be restricted to lumped elastic structural systems. This class of structural systems is distinguished by two main characteristics:

1. The state of stress within any one lumped element is assumed to be fully described by one variable.
2. The displacement pattern of the structure is assumed to be described by the displacement components of a finite number of discrete points.

For any particular load condition the behavior of a lumped elastic structure is completely described by the variables which yield the state of stress in each element of the structure and the displacement components of the pertinent discrete points. These stress and displacement variables may be arranged to form a column matrix for a single load condition.

$$\underset{m \times 1}{\left\{ B_{j1} \right\}}^{3} = \underset{m \times 1}{\left\{ \begin{matrix} \sigma \\ \hline u \end{matrix} \right\}} \tag{1}$$

where m is the total number of behavior variables (stresses and displacements). The behavior variables associated with each of distinct load conditions may be arranged in column matrix form. The behavior matrix is then formed so that each column of this matrix describes the behavior of the structure in the kth load condition.

$$\underset{m \times n}{\left[B_{jk} \right]} = \underset{m \times n}{\left[B_{j1} \; \vdots \; B_{j2} \; \vdots \cdots \vdots \; B_{jn} \right]} \tag{2}$$

The requirements and specification may now be stated concisely as follows:

$$U_{jk} \geq B_{jk} \tag{3}$$

$$B_{jk} \geq L_{jk} \tag{4}$$

where U_{jk} represents the upper limit and L_{jk} represents the lower limit on the jth behavior variable for the kth load condition. Note that this formulation provides for distinct upper and lower limits on each behavior variable for each load condition. The lower limits on stress behavior variables may in general depend upon the design parameters if elastic instability is considered. Let D_p represent the pth design parameter. The cross sectional area of a bar element or the thickness of a shear panel element are, for example, likely design parameters. For various reasons it may be necessary to constrain the value of various design parameters. This requirement can be stated as follows:

$$D_p^{(U)} \geq D_p \tag{5}$$

$$D_p \geq D_p^{(L)} \tag{6}$$

where $D_p^{(U)}$ represents the upper limit on the pth design parameter and $D_p^{(L)}$ represents the lower limit on the pth design paramter. It should be noted that each design parameter D_p, for and structural system,

3. The dimensions of a matrix will be noted below the matrix for clarity when appropriate.

must be equal to or greater than zero.

The governing technology for any lumped elastic structural system may be expressed as a set of m simultaneous equations relating the applied loads to the behavior variables. The set of equations will be made up of equilibrium equations and force-displacement relations. Klein's formulation of the governing technology (6) which is employed in this paper is well adapted to structural synthesis. The governing technology for any lumped elastic structural system subject to n load conditions may be expressed in matrix form as follows:

$$\underset{m \times m}{\left[C_{ij}\right]} \; \underset{m \times n}{\left[B_{jk}\right]} = \underset{m \times n}{\left[A_{ik}\right]} \tag{7}$$

where $\left[B_{jk}\right]$, the behavior matrix, has been defined previously, $\left[A_{ik}\right]$ is the applied loads matrix, and $\left[C_{ij}\right]$ is the configuration matrix. The elements of the matrix $\left[A_{ik}\right]$ are completely defined when the applied load conditions are known. The introduction of known thermal expansions can be readily accomplished since such terms are also elements of the applied loads matrix $\left[A_{ik}\right]$. Hence, a load condition is understood in general to include mechanical as well as thermal loading. The elements of the configuration matrix $\left[C_{ij}\right]$ depend upon that portion of the geometry of the structure that is fixed in advance, the material elastic properties, and the design parameters D_p. It is clear that for any specific design parameter vector

$$\underset{p \times 1}{\left\{D_p\right\}}$$

the behavior of the lumped elastic structural system is given by

$$\left[B_{jk}\right] = \left[C_{ij}\right]^{-1} \left[A_{ik}\right] \tag{7a}$$

This single matrix equation (Eq. 7 or Eq. 7a) embodies the governing technology for the class of structures considered in this paper.

The criterion by means of which choices are to be made between various designs in this paper is selected as minimum weight. The structure with the least total weight which satisfies the complete set of specifications and requirements for all load conditions is considered to have the greatest merit. Let V_r be the volume of the rth element of the structure and let ρ_r be the material weight density of the rth element of the structure. Then the total weight of the structure W may be expressed as follows:

$$W = \underset{1 \times r}{\lfloor V_r \rfloor} \underset{r \times 1}{\left\{\rho_r\right\}} \tag{8}$$

It should be noted that the volume of the rth element is a function of the design parameters D_p. However, there is a large class of problems in which the volume of each element depends linearly on a single design parameter.

The mathematical statement of the most general class of structural synthesis problem considered in this paper may be summarized as follows:

Given $\left\{\rho_r\right\}$ and $\left[A_{ik}\right]$ as well as sufficient geometric and material elastic property information to express $\left[C_{ij}\right]$, $\left[L_{jk}\right]$ $\left[U_{jk}\right]$ and $\lfloor V_r \rfloor$ as functions of the design parameters D_p

Find $\left\{D_p\right\}$ such that $D_p^{(U)} \geq D_p \geq D_p^{(L)}$ and

$$U_{jk} \geq B_{jk}$$

$$B_{jk} \geq L_{jk}$$

and

$$W = \lfloor V_r \rfloor \left\{\rho_r\right\}$$

is a minimum, where the dependence of $\left[B_{jk}\right]$ on D_p is given by

$$\left[B_{jk}\right] = \left[C_{ij}\right]^{-1} \left[A_{ik}\right]$$

FORMULATION OF THE THREE BAR TRUSS PROBLEM

In order to demonstrate the possibility of direct automated structural design the following elementary example embodying the essential features of structural synthesis has been formulated and a program for seeking the optimum balanced design has been written for the IBM 653 digital computer.

A schematic view of the problem considered in detail is shown in Figure 1.

Figure 1 shows a three bar planar truss with its geometric configuration fixed by the given parameters N, β_1, β_2, and β_3. The elastic moduli and the material densities of the three members are designated E_1, E_2, E_3, and ρ_1, ρ_2, ρ_3 respectively. Each member has a constant cross sectional area A_p (p = 1, 2, 3) which is unknown at the outset. The cross sectional areas A_p are the design parameters (D_p) to be determined. The truss is loaded by a given force P_n at a given angle α_n with the x axis as shown in Figure 1. The present program will accommodate any value of n from one to five. In other words, from one to five distinct load conditions each consisting of a specified load P_n at a specified angle α_n can be included in the loading spectrum. There are five behavior variables namely the stress in each of the three bars σ_1, σ_2, σ_3 and the displacement components u_x and u_y of the point s in Figure 1. Thus the kth column of the behavior matrix will be constituted as follows:

$$\left\{ B_{jk} \right\} = \left\{ \begin{array}{c} \sigma_{1k} \\ \sigma_{2k} \\ \sigma_{3k} \\ u_{xk} \\ u_{yk} \end{array} \right\} \tag{9}$$

The upper U_{jk} and lower L_{jk} limits for each element of the behavior matrix B_{jk} are assumed to be given. The only restriction on the design parameters included in the program is

$$A_p \geqslant 0 \qquad p = 1,2,3 \tag{10}$$

The governing technology for the three bar truss consists of two equilibrium equations ($\Sigma F_x = 0$ and $\Sigma F_y = 0$ at point s) and three stress displacement equations (one for each of the three bars). Assuming small displacements the change in length of the pth bar δ_p may be expressed in terms of the displacement components of the point s as follows:

$$\delta_p = -u_x \cos\beta_p - u_y \sin\beta_p \tag{11}$$

The change in length of the pth bar is related to the stress and the temperature rise of the pth bar as follows:

$$\delta_p = \left(\frac{P\ell}{AE}\right)_p + (\bar{\alpha}\Delta T\ell)_p \tag{12}$$

where ℓ_p is the length of the pth bar, α_p is the mean coefficient of thermal expansion for the pth bar; and ΔT_p is the temperature change of the pth bar. Combining Eqs. 11 and 12 and substituting σ_p for $(P/A)_p$ and $N/\sin\beta_p$ for ℓ_p yields

$$\left(\frac{N}{E_p \sin\beta_p}\right)\sigma_p + u_x \cos\beta_p + u_y \sin\beta_p = -\frac{\bar{\alpha}_p \Delta T_p N}{\sin\beta_p} \tag{13}$$

Letting p take on values 1, 2, and 3 in Eq. 13 yields the three pertinent stress displacement relations. The two equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ at point s may be written as follows:

$$\sum_{p=1}^{3} \sigma_p A_p \cos\beta_p = -P_n \cos\alpha_n \tag{14}$$

$$\sum_{p=1}^{3} \sigma_p A_p \sin\beta_p = P_n \sin\alpha_n \tag{15}$$

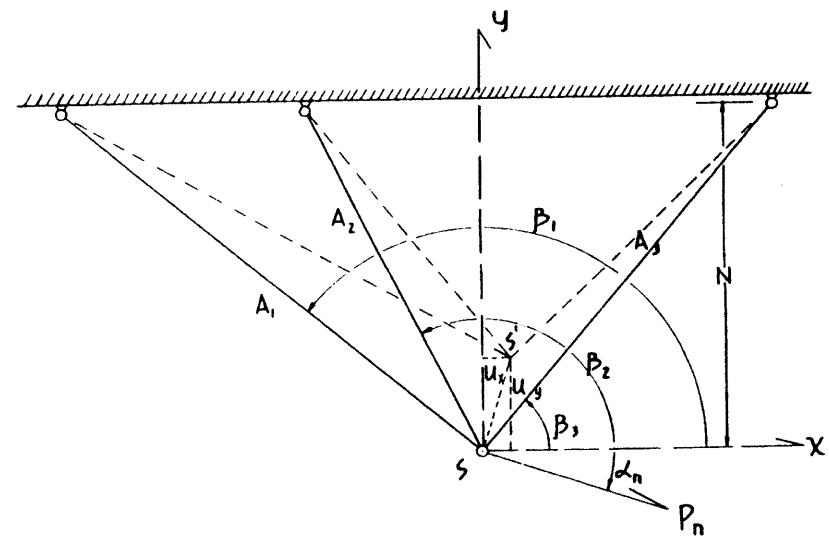

FIG. 1.—THREE BAR TRUSS

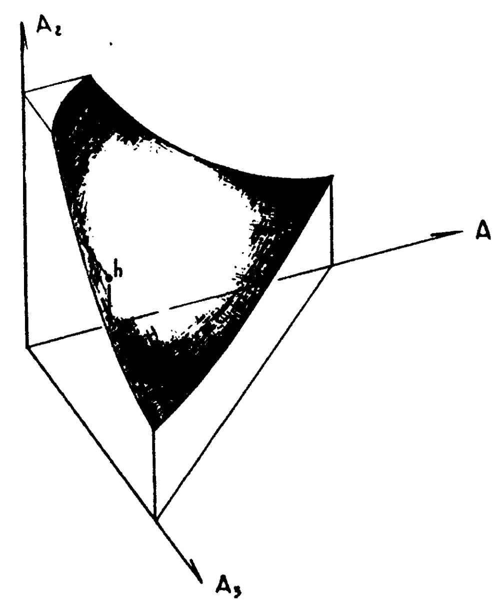

FIG. 2.—A CONCAVE CONSTRAINT SURFACE

The governing technology then may be stated in the matrix form

$$[C_{ij}]\ [B_{jk}] = [A_{ik}]$$

where

$$[C_{ij}] = \begin{bmatrix} A_1\cos\beta_1 & A_2\cos\beta_2 & A_3\cos\beta_3 & 0 & 0 \\ A_1\sin\beta_1 & A_2\sin\beta_2 & A_3\sin\beta_3 & 0 & 0 \\ \frac{N}{E_1\sin\beta_1} & 0 & 0 & \cos\beta_1 & \sin\beta_1 \\ 0 & \frac{N}{E_2\sin\beta_2} & 0 & \cos\beta_2 & \sin\beta_2 \\ 0 & 0 & \frac{N}{E_3\sin\beta_3} & \cos\beta_3 & \sin\beta_3 \end{bmatrix} \quad (16)$$

the kth column of $[B_{jk}]$ is arranged as shown in Eq. 9 and the kth column of $[A_{ik}]$ is constituted as follows:

$$\{A_{ik}\} = \begin{Bmatrix} -P_k\cos\alpha_k \\ P_k\sin\alpha_k \\ -\frac{N\bar{\alpha}_1\ \Delta T_{1k}}{\sin\beta_1} \\ -\frac{N\bar{\alpha}_2\ \Delta T_{2k}}{\sin\beta_2} \\ -\frac{N\bar{\alpha}_3\ \Delta T_{3k}}{\sin\beta_3} \end{Bmatrix}$$

Note that ΔT_{pk} represents the temperature change in the pth bar for the kth load condition.

The three bar truss problem has now been cast in the form of the general structural synthesis problem previously described.

THE METHOD OF ALTERNATE STEPS

Consider a three dimensional space with coordinate axes A_1, A_2, and A_3 (see Figure 2). This space will be referred to as the design parameter space. Any point in the positive octant of this space represents a design of the three bar truss. The upper and lower limitation on each behavior variable in each load condition can be conceived to represent a surface in the design parameter space. This follows from the fact that each element of the behavior matrix is a function of the design parameters A_1, A_2, and A_3. Equating an element of the behavior matrix to its corresponding element in either the upper or lower limitations matrix yields

$$U_{jk} = B_{jk}(A_1, A_2, A_3) \quad (17a)$$

or

$$L_{jk} = B_{jk}(A_1, A_2, A_3) \quad (17b)$$

Such a constraint surface is shown schematically in Figure 2. Points in the design parameter space which are on or above the constraint surface (for example point h Figure 2) satisfy the single constraint represented by the surface while points below the constraint surface violate a requirement and are therefore unacceptable designs. It should be noted that the constraint surface shown in Figure 2 is concave when viewed from the region which is acceptable with respect to this single constraint surface. It has been assumed that the constraint functions are all concave in the design parameter space. In the event that a convexity is encountered the existing digital computer program provides an alarm. No convex constraints have been encountered so far. A proof showing that the constraints are all indeed concave would be most valuable since it would also make it possible to resolve the question of relative minimums.

The total weight of the three bar truss is given by

$$W = N \sum_{p=1}^{3} \frac{\rho A}{\sin\beta}\Big|_p \quad (18)$$

The total weight is seen to be a linear function of the design parameters A_1, A_2, and A_3. Several planes of constant weight are shown in the design parameter space in Figure 3.

Having introduced the notion of a design parameter space, concave constraint surfaces and planes of uniform weight, the method of alternate steps is now described qualitatively. The two dimensional design space shown in Figure 4 is employed solely as an aid in introducing the ideas involved. It will be assumed that an initial trial design which is more than adequate can always be selected. Point 1 in Figure 4 represents a design which more than satisfies all of the constraints G, H, J, and K in Figure 4. Such design points will be referred to as free points, meaning they are acceptable and do not lie on any constraint surface. Note that the constraints K and J in Figure 4 represent minimum member size limitations on A_1 and A_2 respectively. The redesign problem when viewed employing the design parameter space idea reduces to two questions:

(1) which way to go?
(2) how far to go?

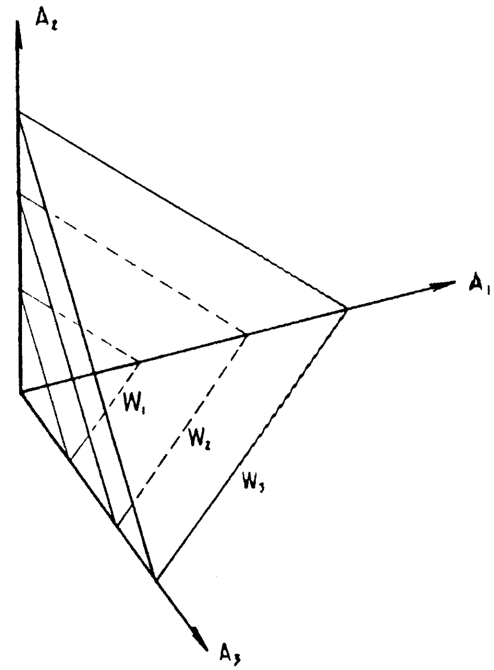

FIG. 3.—PLANES OF CONSTANT WEIGHT

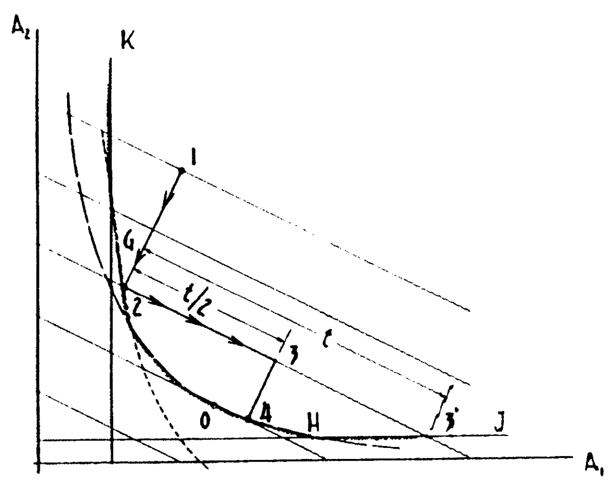

FIG. 4.—TWO DIMENSIONAL DESIGN SPACE

(A) If the current trial design is a free point (such as 1 and 3 in Figure 4) move in the direction of steepest descent until a constraint surface is encountered.

(B) If the current trial design lies on one or more boundaries (such as 2 and 4 in Figure 4) move in a plane of constant weight until a constraint surface is encountered (point 3'), then half the distance of travel t and select the point given by a distance of travel t/2 (point 3) as the next trial design.

The coordinates of the point in the design space representing the (q + 1) th trial design may be expressed in terms of the coordinates of the qth trial design as follows:

$$\underset{3 \times 1}{\left\{A_p^{(q+1)}\right\}} = \underset{3 \times 1}{\left\{A_p^{(q)}\right\}} + t\left\{\phi_p^{(q)}\right\} \tag{19}$$

where the column matrix $\left\{\phi_p^{(q)}\right\}$ specifies the orientation of the line of travel and the scalar t controls the extent and direction of travel.

Examination of the configuration matrix $\left[C_{ij}\right]$ (Eq. 16) reveals that only the elements C_{11} through C_{13} and C_{21} through C_{23} depend upon the design parameters A_p and these elements depend on the A_p linearly. Thus the configuration matrix associated with the (q + 1)th trial design may be expressed in terms of configuration matrix for the qth trial design as follows:

$$\left[C_{ij}^{(q+1)}\right] = \left[C_{ij}^{(q)}\right] + t\left[M_{ij}^{(q)}\right] \tag{20}$$

where

$$\left[M_{ij}^{(q)}\right] = \left[\begin{array}{ccc:cc} \phi_1\cos\beta_1 & \phi_2\cos\beta_2 & \phi_3\cos\beta_3 & 0 & 0 \\ \phi_1\sin\beta_1 & \phi_2\sin\beta_2 & \phi_3\sin\beta_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{21}$$

The matrix $\left[M_{ij}^{(q)}\right]$ may be written in partitioned form as follows:

$$\left[M_{ij}^{(q)}\right] = \begin{bmatrix} \bar{C} & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

where the definition of the 2 x 3 submatrix $[\bar{C}]$ follows from Eq. 21

$$[\bar{C}] = \begin{bmatrix} \phi_1\cos\beta_1 & \phi_2\cos\beta_2 & \phi_3\cos\beta_3 \\ \phi_1\sin\beta_1 & \phi_2\sin\beta_2 & \phi_3\sin\beta_3 \end{bmatrix} \quad (23)$$

When the current (qth) trial design is a free point the orientation of the line of travel is taken normal to the weight planes and

$$\left\{\phi_p^{(q)}\right\} = \left\{\phi_p^{(d)}\right\} = \left\{\frac{\partial W}{\partial A_p}\right\} = \left\{\frac{N\rho_p}{\sin\beta_p}\right\} \quad (24)$$

and these values of ϕ_p when substituted in to Eq. 23 yield the $[\bar{C}]$ matrix associated with travel normal to the weight planes. When the current (qth) trial design lies on one or more constraint surfaces the orientation of the line of travel will be forced to lie in a plane of constant weight. The requirement that the line of travel lie in a plane of constant weight can be stated as follows:

$$\lfloor \phi_p^{(q)} \rfloor \left\{\frac{\partial W}{\partial A_p}\right\} = 0 \quad (25)$$

Equation 25 states that the direction of travel must be orthogonal with the normal to the weight surface. If ϕ_1 and ϕ_2 are now set equal to unity arbitrarily, Eq. 25 can be used to determine ϕ_3. Thus the orientation of a line of travel in a constant weight plane is given by

$$\left\{\phi_p^{(q)}\right\} = \left\{\phi_p^{(a)}\right\} = \left\{\begin{matrix} 1 \\ 1 \\ -\left(\dfrac{\partial W}{\partial A_1} + \dfrac{\partial W}{\partial A_2}\right)\Big/\dfrac{\partial W}{\partial A_3} \end{matrix}\right\} \quad (26)$$

Setting ϕ_2 and ϕ_3 equal to unity and again using Eq. 25 to determine ϕ_1 yields the orientation of a second line of travel in the constant weight plane.

$$\left\{\phi_p^{(q)}\right\} = \left\{\phi_p^{(b)}\right\} = \left\{\begin{matrix} -\left(\dfrac{\partial W}{\partial A_2} + \dfrac{\partial W}{\partial A_3}\right)\Big/\dfrac{\partial W}{\partial A_1} \\ 1 \\ 1 \end{matrix}\right\} \quad (27)$$

Setting ϕ_3 and ϕ_1 equal to unity and using Eq. 25 once again yields the orientation of a third line of travel in the constant weight plane.

$$\left\{\phi_p^{(q)}\right\} = \left\{\phi_p^{(c)}\right\} = \left\{\begin{matrix} 1 \\ -\left(\dfrac{\partial W}{\partial A_1} + \dfrac{\partial W}{\partial A_3}\right)\Big/\dfrac{\partial W}{\partial A_2} \\ 1 \end{matrix}\right\} \quad (28)$$

These three lines of travel in a constant weight plane are shown schematically in Figure 5. It is recognized that situations may arise where these lines of travel will not permit motion to a new trial design even though the current trial design is not the optimum. This difficulty when it arises can be resolved by rotating the directions of travel in the plane of constant weight. Substituting each of the three sets of ϕ_p values (Eqs. 26, 27, and 28) into Eq. 23 yields a matrix $[\bar{C}]$ corresponding to a particular direction of travel $\left\{\phi_p^{(a)}\right\}$, $\left\{\phi_p^{(b)}\right\}$ or $\left\{\phi_p^{(c)}\right\}$ in a constant weight plane.

Having discussed the quantitative formulation of the "which way to go" problem consider now the "how far to go" problem. It will be shown that the behavior matrix at the (q + 1) th trial design point $\left[B^{(q+1)}\right]$ can be expressed as a function solely of the distance of travel t for a selected line of travel eminating from the qth design point. Let the inverse of the configuration matrix at the qth design point (which is known) be partitioned as follows:

$$\underset{5\times5}{\left[C_{ij}^{(q)}\right]^{-1}} = \left[\begin{array}{c|c} \underset{3\times2}{D} & \underset{3\times3}{G} \\ \hline \underset{2\times2}{H} & \underset{2\times3}{K} \end{array}\right] \quad (29)$$

Following the procedure as given in (13) it can be shown that

$$\underset{5\times5}{\left[C_{ij}^{(q+1)}\right]^{-1}} = \underset{5\times5}{\left[C_{ij}^{(q)}\right]^{-1}} \underset{5\times5}{\left[Q\right]^{-1}} \quad (30)$$

where

$$[Q]^{-1} = \left[\begin{array}{c|c} \underset{2\times2}{[I + tR]^{-1}} & \underset{2\times3}{-t\,[I + tR]^{-1}[\bar{C}][G]} \\ \hline \underset{3\times2}{0} & \underset{3\times3}{I} \end{array}\right] \quad (31)$$

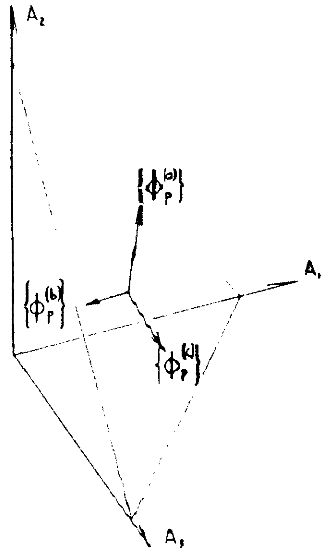

FIG. 5.—LINES OF TRAVEL IN CONSTANT WEIGHT PLANE

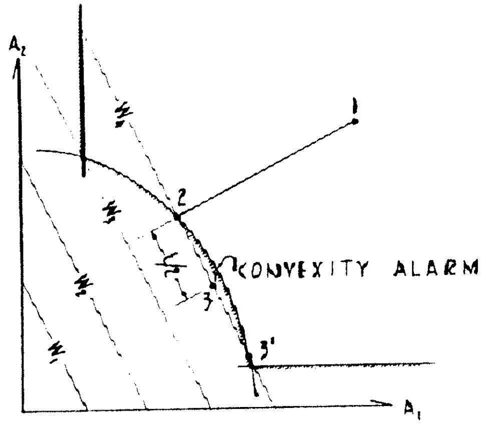

FIG. 6.—SCHEMATIC OF CONVEXITY ALARM CONDITION

and

$$\underset{2x2}{[R]} = \underset{2x3}{[\bar{C}]}\ \underset{3x2}{[D]} \tag{32}$$

It can also be shown that

$$[I + tR]^{-1} = 1/g(t)\ [I + t\bar{R}] \tag{33}$$

where

$$[\bar{R}] = \begin{bmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{bmatrix} \tag{34}$$

and

$$g(t) = (R_{11}R_{22} - R_{12}R_{21})\,t^2 + (R_{11} + R_{22})t + 1 \tag{35}$$

Now it follows that

$$[Q]^{-1} = 1/g(t)\ [I] + t/g(t)\ [S] + t^2/g(t)[\bar{S}] \tag{36}$$

where

$$[S] = \left[\begin{array}{c|c} \underset{2x2}{[\bar{R}]} & -\underset{2x3}{[\bar{C}][a]} \\ \hline \underset{3x2}{0} & \underset{3x3}{(R_{11} + R_{22})\,[I]} \end{array}\right] \tag{37}$$

and

$$[\bar{S}] = \left[\begin{array}{c|c} \underset{2x2}{[0]} & -\underset{2x3}{[\bar{R}][\bar{C}][a]} \\ \hline \underset{3x2}{[0]} & \underset{3x3}{(R_{11}R_{22} - R_{12}R_{21})\,[I]} \end{array}\right] \tag{38}$$

The applied loads matrix $[A_{jk}]$ is independent of the change in the design parameters A_p therefore the behavior matrix at the $(q + 1)$ th trial design point $[B_{jk}^{(q+1)}]$ may be expressed as follows:

$$\underset{5xn}{[B^{(q+1)}]} = \underset{5x5}{[C^{(q+1)}]^{-1}}\ \underset{5xn}{[A]} = \underset{5x5}{[C^{(q)}]^{-1}}\ \underset{5x5}{[Q]^{-1}}\ \underset{5xn}{[A]} \tag{39}$$

Substituting for $[Q]^{-1}$ from Eq. 36 and recalling that $[C^{(q)}]^{-1}\ [A] = [B^{(q)}]$ yields

$$\underset{5xn}{[B^{(q+1)}]} = 1/g(t)\,\underset{5xn}{[B^{(q)}]} + t/g(t)\,\underset{5xn}{[T]} + t^2/g(t)\,\underset{5xn}{[\bar{T}]} \tag{40}$$

where

$$\underset{5xn}{[T]} = \underset{5x5}{[C^{(q)}]^{-1}}\ \underset{5x5}{[S]}\,\underset{5xn}{[A]} \tag{41}$$

and

$$\underset{5xn}{[\bar{T}]}\quad \underset{5x5}{[C^{(q)}]^{-1}}\ \underset{5x5}{[\bar{S}]}\,\underset{5xn}{[A]} \tag{42}$$

Equation 40 gives an expression for each element of the behavior matrix as a function of the distance of travel t. Equating this expression for an element of the behavior matrix (B_{jk}) to the corresponding upper (U_{jk}) or lower (L_{jk}) limit matrix element yields quadratic equations of the following form:

$$\left[(R_{11}R_{22} - R_{12}R_{21})U_{jk} - \bar{T}_{jk}\right]t^2 + \left[(R_{11} + R_{22})U_{jk} - T_{jk}\right]t + (U_{jk} - B_{jk}) = 0 \tag{43}$$

$$\left[(R_{11}R_{22} - R_{12}R_{21})L_{jk} - \bar{T}_{jk}\right]t^2 + \left[(R_{11} + R_{22})L_{jk} - T_{jk}\right]t + (L_{jk} - B_{jk}) = 0 \tag{44}$$

Imaginary roots resulting from the solution of Eq. 43 or Eq. 44 are neglected while real roots represent the distance of travel from the current trial design point (q) to a constraint surface. The first root t which meets the following set of requirements is selected and used to establish the (q + 1) th trial design:

1. Each element of the behavior matrix (see Eq. 40) must satisfy all of the constraints within a tolerance ϵ that is

$$(U_{jk} - B_{jk}) \geqq -\epsilon \tag{45}$$

and

$$(B_{jk} - L_{jk}) \geqq -\epsilon \tag{46}$$

2. The value of t must not result in values of $A_p^{(q+1)}$ that are less than the lower limit prescribed, that is

$$t \geqq \frac{(A_p)_{min} - A_p^{(q)}}{\phi_p^{(q)}} \tag{47}$$

3. The value of t must result in a significant change (δ) in at least one of the design parameters

$$\left| A_p^{(q+1)} - A_p^{(q)} \right| = \left| \phi_p^{(q)} \, t \right| \geqq \delta \tag{48}$$

Searching sequentially through all of the possible values of t yielded by Eq. 43 for each element U_{jk} and then through all of the possible values of t yielded by Eq. 44 for each element L_{jk} usually yields a value of t which satisfies the three requirements stated above. If the current trial design lies on one or more boundaries and the search for a t satisfying the requirements stated above fails, it is possible to continue the search for a satisfactory t using a different direction of travel in the

plane of constant weight. The manner in which a selected t is used to determine the (q + 1) th trial design depends upon whether the current trial design is a free point or not. If the current trial design is a free point then the orientation of the line of travel is normal to the planes of constant weight and the new trial is determined as follows:

$$\left\{A_p^{(q+1)}\right\} = \left\{A_p^{(q)}\right\} + t^* \left\{\phi_p^{(d)}\right\} \tag{49}$$

where

t^* = the selected value of t satisfying all requirements. If the current trial design is not a free point then it must lie on one or more constraint surfaces. The orientation of the line of travel lies in a plane of constant weight and the new trial design is determined as follows:

$$\left\{A_p^{(q+1)}\right\} = \left\{A_p^{(q)}\right\} + t^*/2 \left\{\phi_p^{(q)}\right\} \tag{50}$$

where $\left\{\phi_p^{(q)}\right\}$ is given by Eq. 26, or Eq. 27 or Eq. 28. If a convex constraint surface exists, Eq. 50 could lead to the selection of a trial design which is in violation of some of the constraints. This situation is shown schematically in a two dimensional design parameter space in Figure 6. An alarm which will detect this occurrence is provided in the computer program.

To sum up then the method of alternate steps is a technique for seeking the minimum weight balanced design. Whenever possible redesign takes place so as to reduce the total weight at the greatest possible rate (steepest descent). When this is not possible, redesign takes place maintaining the total weight constant.

THE COMPUTER PROGRAM

The computer program for the structural synthesis of the three bar truss system based on the method of alternate steps was written for the IBM 653 digital computer at the Case Computing Center using Runcible Compiler language. Because several special subroutines were used to save space, the program was compiled using multipass operation. That is to say the Runcible compiler was employed to prepare a program in SOAP language and the SOAP program was then subsequently used to obtain a machine language program. It should be emphasized that the potential value of the structural synthesis concept depends upon the availability of an ever increasing computer capability. The existing computer program for the three bar truss requires 1939 locations of the 2000 available on the magnetic drum. While operating times are difficult to predict because of the nature of the procedure employed, less than 30 minutes has usually been sufficient to obtain an optimum design that is within 1% of the minimum possible total weight for cases with two or three independent load conditions.

Figure 7 shows a block diagram outlining the major phases of the program. While this block diagram is adequate for grasping the overall problem a much more detailed block diagram was, of course, employed as a guide to the actual programming and coding.

NUMERICAL RESULTS

Numerical results for two cases will be presented in detail. The input data for the first case follows:

Case I Input.—

$\beta_1 = 135°$ $\beta_2 = 90°$ $\beta_3 = 45°$

$P_1 = 30$ $\alpha_1 = 60°$

$P_2 = 20$ $\alpha_2 = 180°$

$N = 1$

$\rho_1 = \rho_2 = \rho_3 = 1$

$E = 1$[4]

$n = 2$ (number of load conditions)

$$\left[U_{jk}\right] = \begin{bmatrix} 20 & 20 \\ 20 & 20 \\ 20 & 20 \\ 200 & 200 \\ 200 & 200 \end{bmatrix} \qquad \left[L_{jk}\right] = \begin{bmatrix} -15 & -15 \\ -15 & -15 \\ -15 & -15 \\ -150 & -150 \\ -150 & -150 \end{bmatrix}$$

$$\left\{A_p^{(1)}\right\} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix} \qquad A_p \geq 0$$

$$\epsilon = .01 \qquad \delta = .001$$

Note that the displacement limitations have been set high so that they are not active. The stress limitations will thus prove to be critical in these examples. The results of the synthesis are presented in Table I. The behavior matrix based on the final design shown in Table I for Case I is

$$\left[B_{jk}\right] = \begin{bmatrix} 19.863 & -14.993^* \\ 19.984^* & +5.003 \\ +1.206 & +19.996^* \\ +19.743 & -34.988 \\ -19.984 & -5.003 \end{bmatrix} \tag{51}$$

4. Setting E = 1 in effect means that the displacement u_x and u_y are replaced by (u_x E) and (u_y E). (See Eq. 16)

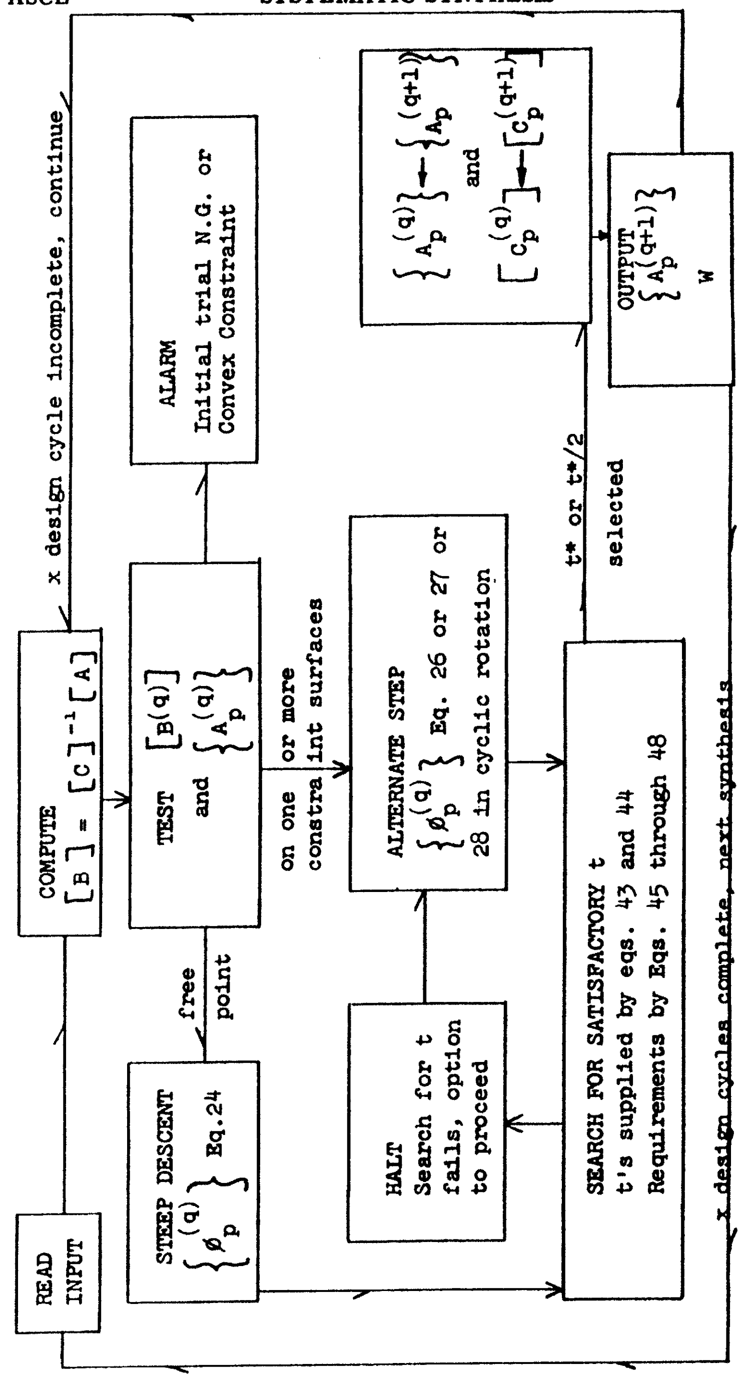

FIG. 7.—FLOW DIAGRAM

It is interesting to note that in this case the optimum balanced design is a fully stressed design in the sense defined in (14). This is to say each member is essentially fully stressed in at least one load condition. (See values in Eq. 51 marked with *.)

Figure 8 represents the constrained weight surface for Case I. The actual surface is continuous, but it has discontinuities in gradient which are not shown in Figure 8. Every design point on or above the surface shown in Figure 8 satisfies all of the requirements imposed on the three bar truss in Case I. Several of the trial designs listed in Table I including the final minimum weight design are plotted in Figure 8.

The input data for the second example follows:

Case II Input.—

$\beta_1 = 135°$ $\qquad \beta_2 = 90°$ $\qquad \beta_3 = 45°$

$P_1 = 40$ $\qquad \alpha_1 = 45°$

$P_2 = 30$ $\qquad \alpha_2 = 90°$

$P_3 = 20$ $\qquad \alpha_3 = 135°$

$N = 1$

$\rho_1 = \rho_2 = \rho_3 = 1$

$E = 1$

$n = 3$ (number of load conditions)

$$\left[U_{jk}\right] = \begin{bmatrix} 5 & 5 & 5 \\ 20 & 20 & 20 \\ 5 & 5 & 5 \\ 200 & 200 & 200 \\ 200 & 200 & 200 \end{bmatrix} \qquad \left[L_{jk}\right] = \begin{bmatrix} -5 & -5 & -5 \\ -20 & -20 & -20 \\ -5 & -5 & -5 \\ -200 & -200 & -200 \\ -200 & -200 & -200 \end{bmatrix}$$

$$\left\{A_p^{(1)}\right\} = \begin{Bmatrix} 8.0 \\ 2.4 \\ 3.2 \end{Bmatrix} \qquad A_p \geqq 0$$

$$\epsilon = 0.01 \qquad \delta = .001$$

The results of the synthesis are presented in Table II. The behavior matrix based on the final design shown in Table II for Case II is

$$\left[B_{jk}\right] = \begin{bmatrix} 5.000^* & 1.827 & -0.777 \\ 3.445 & 6.305 & 4.222 \\ -1.555 & 4.478 & 4.999^* \\ 6.555 & -2.651 & -5.776 \\ -3.445 & -6.305 & -4.222 \end{bmatrix}$$

TABLE I.—SUMMARY OF RESULTS CASE I

Cycle	A_1	A_2	A_3	W	A_2/A_1	A_3/A_1
1	1.000	1.000	1.000	3.828	1.000	1.000
2	0.943	0.960	0.943	3.626	1.018	1.000
3	1.048	1.064	0.764	3.626	1.015	0.729
4	1.037	1.057	0.753	3.589	1.019	0.726
5	1.259	0.927	0.624	3.589	0.736	0.496
6	1.180	0.872	0.545	3.311	0.739	0.462
7	1.261	0.642	0.626	3.311	0.509	0.496
8	1.211	0.606	0.576	3.132	0.500	0.476
9	1.166	0.561	0.652	3.132	0.481	0.559
10	1.118	0.527	0.604	2.962	0.471	0.540
11	1.089	0.544	0.622	2.962	0.500	0.571
12	1.081	0.539	0.614	2.936	0.499	0.568
13	1.078	0.546	0.611	2.936	0.506	0.567
14	1.077	0.545	0.610	2.930	0.506	0.566
18	1.072	0.544	0.611	2.924	0.507	0.570

TABLE II.—SUMMARY OF RESULTS CASE II

Cycle	A_1	A_2	A_3	W	A_2/A_1	A_3/A_1
1	8.000	2.400	3.200	18.237	0.300	0.400
2	7.563	2.091	2.763	16.695	0.276	0.365
3	7.343	1.871	3.138	16.695	0.255	0.427
4	7.156	1.739	2.951	16.031	0.243	0.412
5	7.140	1.748	2.960	16.031	0.245	0.415
13	7.099	1.849	2.897	15.986	0.260	0.408

It is enlightening to observe that in this case the optimum balanced design is not a fully stressed design in the sense defined in (14). Member 2 is never fully stressed yet an effort to reduce its weight will lead to a net weight increase. Figure 9 represents the constrained weight surface for Case II. A few of the trial designs listed in Table II are plotted in Figure 9.

The abbreviated results for two additional examples are also given.

Case III Input. –

$\beta_1 = 135°$ $\quad \beta_2 = 90°$ $\quad \beta_3 = 45°$

$P_1 = 20$ $\quad \alpha_1 = 45°$

$P_2 = 15$ $\quad \alpha_2 = 90°$

$P_3 = 10$ $\quad \alpha_3 = 135°$

$N = 1$

$\rho_1 = \rho_2 = \rho_3 = 1$

$E = 1$

$n = 3$ (number of load conditions)

$$[U_{jk}] = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \\ 100 & 100 & 100 \\ 100 & 100 & 100 \end{bmatrix} \qquad [L_{jk}] = \begin{bmatrix} -10 & -10 & -10 \\ -10 & -10 & -10 \\ -10 & -10 & -10 \\ -100 & -100 & -100 \\ -100 & -100 & -100 \end{bmatrix}$$

$$\left\{A_p^{(1)}\right\} = \begin{matrix} 2.0 \\ 2.0 \\ 2.0 \end{matrix} \qquad A_p \geq 0$$

$\epsilon = 0.01$ $\qquad \delta = 0.001$

Case III Output - Cycle 20

$A_1 = 1.707$ $\quad A_2 = 0.940$ $\quad A_3 = 0.526$

$W = 4.099$

The behavior matrix based on this design is

$$[B_{jk}] = \begin{bmatrix} 9.996^* & 2.342 & -2.790 \\ 4.416 & 9.940^* & 7.164 \\ -5.580 & 7.598 & 9.953^* \\ 15.576 & -5.256 & -12.743 \\ -4.416 & -9.940 & -7.163 \end{bmatrix}$$

It is noted that in this case the minimum weight balanced design happens to be a fully stressed design.

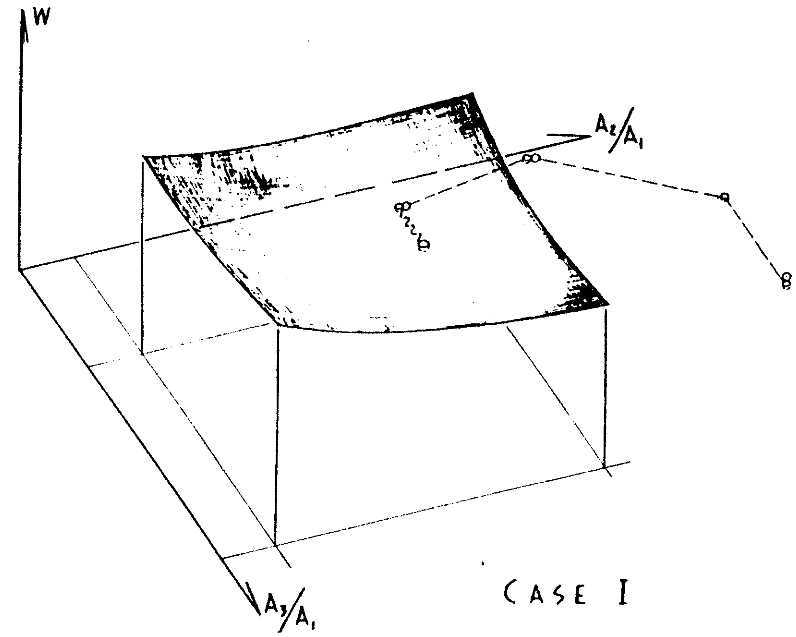

FIG. 8.–CONSTRAINED WEIGHT SURFACE

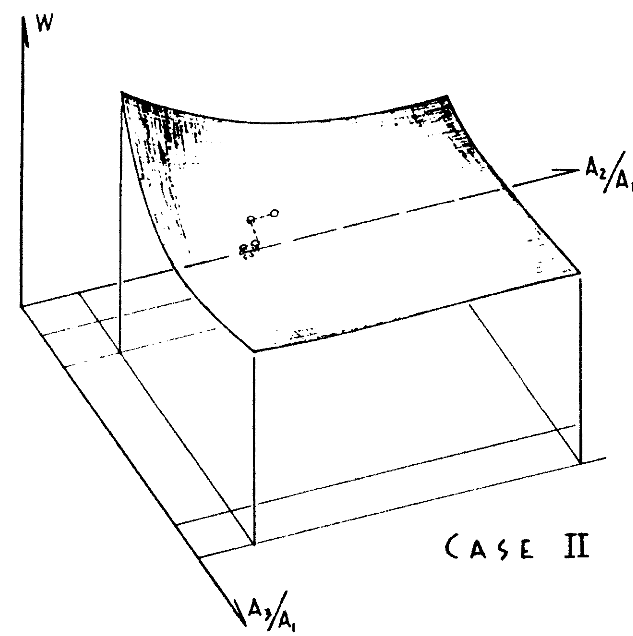

FIG. 9.–CONSTRAINED WEIGHT SURFACE

Case IV Input.—

$\beta_1 = 135°$ $\beta_2 = 90°$ $\beta_3 = 45°$

$P_1 = 20$ $\alpha_1 = 45°$

$P_2 = 20$ $\alpha_2 = 135°$

$N = 1$

$\rho_1 = \rho_2 = \rho_3 = 1$

$E = 1$

$n = 2$ (number of load conditions)

$$\left[U_{jk}\right] = \begin{bmatrix} 20 & 20 \\ 20 & 20 \\ 20 & 20 \\ 200 & 200 \\ 200 & 200 \end{bmatrix} \qquad L_{jk} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \\ -15 & -15 \\ -150 & -150 \\ -150 & -150 \end{bmatrix}$$

$$\left\{A_p^{(1)}\right\} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix} \qquad A_p \geq 0$$

$\epsilon = 0.01$ $\delta = 0.001$

Case IV Output - Cycle 4

$A_1 = 0.784$ $A_2 = 0.422$ $A_3 = 0.784$

$W = 2.639$

The behavior matrix based on this design is

$$\left[B_{jk}\right] = \begin{bmatrix} 20.000^* & -5.513 \\ 14.487 & 14.487 \\ -5.513 & 20.000^* \\ 25.513 & -25.513 \\ -14.487 & -14.487 \end{bmatrix}$$

In this case the minimum weight balanced design does not happen to be fully stressed.

CONCLUSIONS

The general problem of structural synthesis for lumped elastic systems has been formulated. The synthesis of a three bar truss, which is an elementary example exhibiting the important characteristics of this larger class of structures, has been programmed for the IBM 653 digital computer. A procedure designated the method of alternate steps is employed to carry out the synthesis.

The numerical results point out that the minimum weight balanced design for a statically indeterminate structure is not necessarily one in which each member is fully utilized in at least one load condition. Viewing the synthesis problem using the concept of a design parameter space, it follows that a design in which each member is fully utilized in at least one load condition must lie at the intersection of p constraint surfaces in the design parameter space (i.e. corners). There is, however, no reason why such corner points should necessarily be points representing designs of minimum weight.

In concluding, it seems appropriate to reflect for a moment on the overall structural design problem. Just as structural analysis must be viewed as a component part of structural synthesis, so must the methods of synthesis be viewed in proper perspective. The overall formulation of structural design problems past, present, and future presents a substantial challenge. Particularly the matter of conceiving and stating more realistic design philosophies must be given more attention. Within the limits of what is discussed herein, structural synthesis is not intended, nor indeed is it capable of being, a substitute for creativity. The demands o made on the structural engineer will continue to require the conceiving of new configurations. The need for experience, judgment, and ingenuity can be expected to be higher than ever before. However, structural synthesis shows considerable promise and further developments along these lines should prove to be a useful scientific aid in structural design.

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