An Analytical Directional Growth Topology Parameterization to Enforce Manufacturing Requirements

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A new method to generate castable or extrudable optimal topology designs in a given direction is presented. The method is based on parameterizing the design domain and is suitable to be used with gradient-based topology optimization methods. The development was motivated by the need for generating structural design proposals that could be manufactured with minimum changes. The proposed method takes into consideration irregular meshes commonly used in industrial applications. It also takes into consideration minimum member sizes that are needed for manufacturability and to control the checkerboard phenomena. The method is discipline independent and was tested on linear static and dynamic problems. The method is very efficient as it reduces the number of design variables and does not need to add additional constraints to reflect the manufacturing requirements. The manufacturing requirements are built in a parameterization of the design variables. The proposed method was implemented in the GENESIS program and examples that show its effectiveness are included.

Nomenclature

A^e_i	=	area density of element i
A^g_j	=	area density of grid j
A^p_k	=	area density of pole k
D_{jk}	=	radial distance between grid j and pole k
ρ^e_i	=	volume fraction of element i
F	=	objective function
g_j	=	jth constraint
h	=	discretization parameter
h^e_i max	=	maximum height of element i
h^e_i min	=	minimum height of element i
H^e_i	=	material growth associated to element i
H^e_i max	=	maximum height of the design domain on the vicinity of element i
H^e_i min	=	minimum height of the design domain on the vicinity of element i
i	=	element index
j	=	grid index or constraint index when used with gj
k	=	pole index
l^e_i	=	element growth fraction of element i
L^e_i	=	growth fraction at element i
L^g_j	=	growth fraction at grid j
L^p_k	=	growth fraction at pole k
W_{jk}	=	weighting factor
Ω	=	design domain
Ω_{xy}	=	parametric design domain

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I. Introduction

The field of topology optimization has been extensively developed and has successfully addressed a variety of problems encountered by researchers and practitioners. However, little attention has been given on how to develop a topology optimization capability that produces design proposals that are consistent with certain manufacturing processes, such as casting or extrusion, where no interior cavities are allowed. Consequently, designers using topology optimization have encountered difficulties interpreting results and adapting them to their final designs. This has motivated the study of how to add this capability to the GENESIS program[1]. In researching the literature, only few papers can be found that deals with this subject. Harzheim and co-workers, from Adams Opel AG, appears to be the first ones dealing with methods that attempt to incorporate casting requirement in topology optimization[2]. They developed a "biological" growth rule based on stress levels in a prescribed direction and added it on the TopShape program. The TopShape program is a derivative of the SKO (Soft Kill Optimization) and CAO (Shape Optimization) programs and seems to work well for this task. However, since the "biological" growth rule is not based on a differentiable analytical expression, it is not usable for programs like GENESIS that uses gradient information to perform optimization. Another drawback of the biological rule implemented in TopShape is that it requires the finite element model to use a voxel mesh (regular mesh build with hexahedral elements).

Zhou and co-workers, proposed an alternative method for optimizing parts with casting requirements. Their proposed method is suitable for non-linear programming based topology optimization programs[3]. Their approach consists of creating a series of constraints that essentially forces the densities of element in "lower" rows to be higher than the densities of elements in "higher" rows. In other words, their approach creates mathematical constraints to enforce the manufacturing requirements. Although this second approach appears to work, the authors of this paper predict that their approach may be unnecessarily expensive. Why allow design freedom that is only going to be constrained away?

In an attempt to overcome the limitations of existing methods, the authors of this paper set out to devise a method that has the following characteristics:

1. Be response independent: Do not rely on growth rules that are functions of stress levels, as they cannot be applied to problems without stress, e.g., problems with natural frequency constraints.
2. Be general enough to be used with both regular and irregular finite element meshes.
3. Be efficient: Avoid unnecessarily increasing the size of the optimization problem.

This paper presents a new method that appears to satisfy the above requirements and will be discussed after giving a brief description of the general topology optimization problem.

II. The Optimization Problem

The topology optimization problem of interest, without considering manufacturing requirements, is shown next:

$$\begin{aligned} & Min\, F(\rho_1, \rho_2, \ldots, \rho_n) \\ & \text{such that :} \\ & g_j(\rho_1, \rho_2, \ldots, \rho_n) \le 0; \quad j = 1, m \\ & 0.0 \le \rho_i \le 1.0; \quad i = 1, n \end{aligned} \tag{1}$$

where F is the objective function, g_j are the constraints and ρ_i are the design variables that represents element volume fractions

This topology optimization problem is solved in GENESIS using a sequence of approximate problems based on the approximation concept approach. Each approximate problem contains approximate functions and is solved using the general-purpose optimizer BIGDOT. The material properties are updated using predefined rules (Power, Reuss, Voigt and/or combination of the last two) that help to get at 0-1 solutions that represent the void or solid state of each of the designable elements in the finite element model.

Approximation concepts for traditional structural optimization (sizing) were introduced by Schmit et al., in the mid-seventies[4,5]. Efficient ways to approximate functions are discussed on references 6 through 13. The idea to design a Young's modulus with a predefined rule for the purpose of creating voids that represent a topological design was presented by Bendsoe et al. in the late eighties[14]. The Reuss and Voigt rules are discussed in Ref. 15. The BIGDOT program is a large-scale non-linear exterior penalty based optimizer developed by Vanderplaats[16]. Details on how these key ideas and others are implemented in the GENESIS program can be found in Ref. 17.

III. Design Variables

As topology optimization is used to solve for the material distribution problem, the traditional approach has been to select the element volume fractions as the design variables. The element volume fraction is defined as follows:

$$\rho_i = \frac{V_i^e}{V0_i^e} \tag{2}$$

where V_i^e is the solid fraction of volume in the element i and $V0_i^e$ is the total volume of element i.

IV. The Growth Direction

The growth direction is the direction where the structure will be allowed to grow.

Figure 1: The growth direction

The arrow pointing upward on the right of Fig. 1 represents a pre-selected growth direction. The left part of Fig. 1, also shows the hypothetical initial design of a structure before optimization. The gray color indicates it has a uniform distributed material that is neither fully solid nor fully void. The center shows a typical topology optimization result that maximizes the stiffness of the structure, subject to a torsional load. The black color denotes a fully solid state and the white in the center a fully developed hole. The U shaped structure on the right part shows one desirable topology result, which is not an optimum from the stiffness point of view, but it can be manufactured using casting techniques

V. Design Variables Parameterization

The key idea of this work is to parameterize the element volume fractions with parameters that explicitly take into consideration the growth direction. The relationship between the element volume fraction and the elements parameters is shown next:

$$\rho_i = A_i^e * \ell_i^e \tag{3}$$

where A_i^e is the element area density and ℓ_i^e is the element growth fraction.

The element area density is the volume fraction of the element when it is completely filled. A_i^e is allowed to take values between zero and one just like the element volume fraction. The element growth fraction is used to measure if the element is completely filled with material (ℓ_i^e =1.0), is partially filled with material (0.0< ℓ_i^e <1.0) or not filled at all (ℓ_i^e = 0). The element growth fraction is measured in the direction of the filled material:

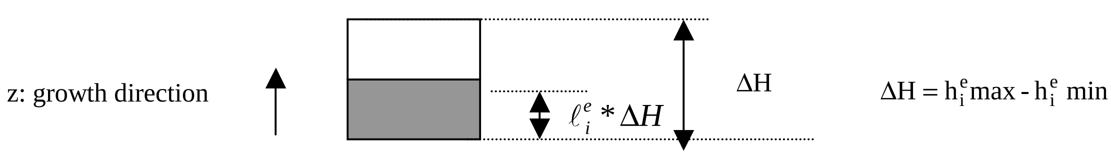


Figure 2: The element growth fraction

In Fig. 2, $h^e_i max$ and $h^e_i min$ are respectively the highest and lowest points of the element measured in the growth direction. These quantities can be easily calculated comparing the height of all grids of the element. The shape of the individual element can be uniform as shown in Fig. 2 or non-uniform such a tetrahedral with 4 different faces.

A. Pole Parameters

A coordinate system that has its z-axis parallel to the growth direction is used. All grids are transformed to that coordinate system to simplify calculations. A plane that is perpendicular to z is used as a reference plane. The projection of the design domain Ω into the XY plane is defined as the parametric domain Ω_{xy}, which is discretized using a constant spacing h. Lines parallel to the growth direction that start on the discrete points of Ω_{xy} are named poles. The total number of poles is NP.

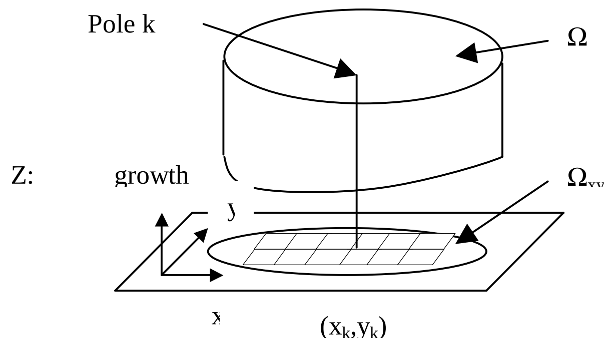


Figure 3. The Pole Parameters

For every pole in Ω_{xy}, two parameters are used: A^p_k and L^p_k. The first parameter is named pole area density and is constructed to represent the density in the vicinity of the pole *k*. The second parameter, L^p_k, is named pole growth fraction and represents the fraction of the height that is filled with material in the vicinity of the pole *k*.

The pole area density and the pole growth fraction parameters can take values between 0.0 and 1.0.

$$\left.\begin{array}{l} 0.0 \le A^p_k \le 1.0 \\ 0.0 \le L^p_k \le 1.0 \end{array}\right\} \tag{4}$$

For convenience the parameters can be stored as vectors:

$$\begin{array}{l} \vec{A} = (A^p_1, A^p_2, .., A^p_{NP}) \\ \vec{L} = (L^p_1, L^p_2, ..., L^p_{NP}) \end{array} \tag{5}$$

The desired parameterization is constructed so that when all pole parameters are 1.0, all element volume fractions are 1.0. The following sections describe a procedure to achieve this.

B. Grid Parameters

The area densities and the growth fraction parameters are calculated at the grids as weighted averages of their corresponding pole parameters as shown next:

$$\left.\begin{array}{l} A^g_j = \sum_{k=1}^{NPj} w_{jk}(x_j, y_j) A^p_k \\ L^g_j = \sum_{k=1}^{NPj} * w_{jk}(x_j, y_j) L^p_k \end{array}\right\} \tag{6}$$

Where:

$$w_{jk}(x_j, y_j) = \frac{e^{-\alpha(\frac{D_{jk}}{s})^2}}{\sum_{k=1}^{NPj} e^{-\alpha(\frac{D_{jk}}{s})^2}} \tag{7}$$

$$D_{jk}(x_j, y_j) = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2} \tag{8}$$

In the above equations w_{jk} is a Gaussian weighting factor similar to the ones used in meshless methods[18]. The weighting factors are normalized so that when all pole parameters are 1.0, the corresponding grid parameters are 1.0. In the exponential expression, s and α are predefined values used to accelerate or reduce the influence of the pole on the grids; s and α, along with the h parameter, are also used to impose a minimum member size. D_{jk} is the radial distance from grid j to pole k. As these expressions are z independent, they essentially enforce a constant function value over z. It should be noticed that the weighting factors with small values are discarded and that the equations associated to them are restricted to work only with the relevant weighting factors.

C. Element Parameters

The area density and the growth fraction evaluated at the center of the element i can be calculated as a simple average of the grids that define the element:

$$\left.\begin{aligned} A_i^e &= \sum_{j=1}^{NGi} \frac{A_j^g}{NGi} \\ L_i^e &= \sum_{j=1}^{NGi} \frac{L_j^g}{NGi} \end{aligned}\right\} \tag{9}$$

NGi is the number of the grids associated with ith element. This averaging is intended to help to get a better approximation at the center of the element on coarse meshes. An alternative and simpler scheme would be to evaluate these parameters at the center of gravity of the element.

D. Interpolation Functions

Combining Eqs. (6), (7), (8), and (9) gives the following interpolation functions:

$$\left.\begin{aligned} A_i^e &= \Psi_i\left(\vec{A}\right) \\ L_i^e &= \Psi_i\left(\vec{L}\right) \end{aligned}\right\} \tag{10}$$

Where

$$\left.\begin{aligned} \Psi_i(\vec{A}) &= \sum_{j=1}^{NGi} \frac{1}{NGi}\left(\sum_{k=1}^{NP} W_{jk}(x_j, y_j) A_k^p\right) \\ \Psi_i(\vec{L}) &= \sum_{j=1}^{NGi} \frac{1}{NGi}\left(\sum_{k=1}^{NP} W_{jk}(x_j, y_j) L_k^p\right) \end{aligned}\right\} \tag{11}$$

E. Analytical Growth Rule

Having calculated the growth fraction L^e_i of the element i, one can calculate the actual material growth associated to that element:

$$H_i^e(L_i^e) = (H_i^e \max - H_i^e \min) * L_i^e + H_i^e \min \tag{12}$$

In the preceding equation, $H_i^e \max$ and $H_i^e \min$ correspond to characteristic maximum and minimum heights of the design domain in the vicinity of the element i. The program calculates these quantities automatically.

From the actual material growth one can calculate the element growth fraction using the following expression:

$$\ell_i^e = \Phi(H_i^e) = \begin{cases} \phi(H_i^e) & \text{if } h_i^e \min \le H_i^e \le h_i^e \max \\ 1.0 & H_i^e \ge h_i^e \max \\ 0.0 & H_i^e \le h_i^e \min \end{cases} \tag{13}$$

Where:

$$\phi(H_i^e) = \frac{H_i^e - h_i^e \min}{h_i^e \max - h_i^e \min} \tag{14}$$

In the above equations $h_i^e\max$ and $h_i^e\min$ correspond to the highest and lowest point in the design domain in the vicinity of element *i*. Equation (13) represents a rule for filling the element linearly. This approximation could be replaced with a more accurate one, but it is deemed to be adequate enough as its only purpose is to achieve a 0-1 solution where it is sufficient to know if the element is completed filled or not. This growth rule looks similar to the one used in TopShape, but it uses the H_i^e variable that is a explicit function of the pole parametric variables. The rule provides a way to analytically describe the direction of filling. By contrast, the TopShape growth rule is a function of the stress at the element.

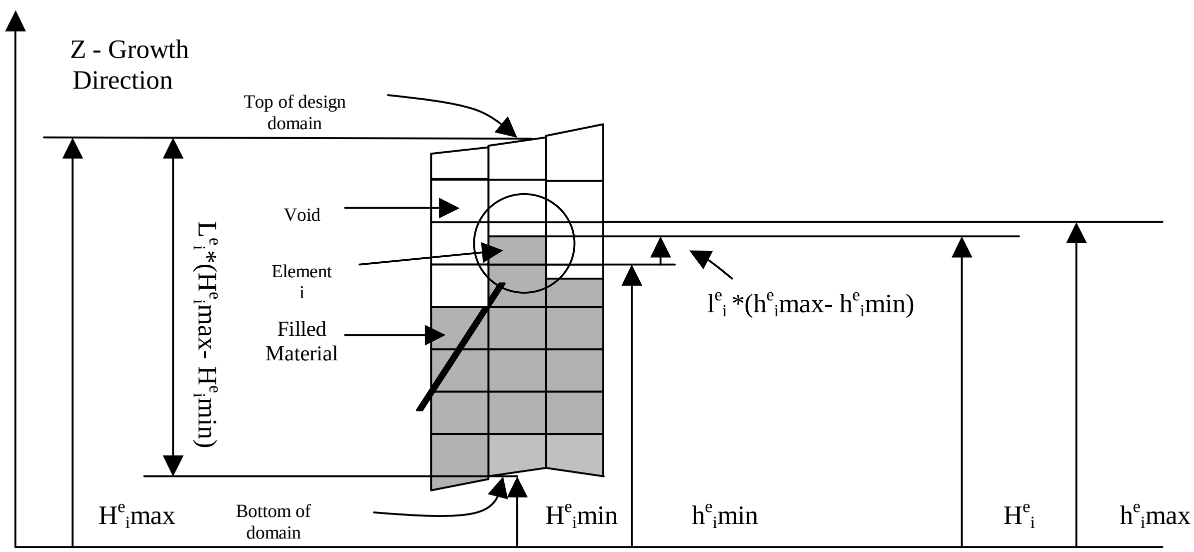


Figure 4. The Growth Parameters

By combining Eqs. (10), (12), and (13), we can construct the equation that links the pole growth fraction with the element growth fraction:

$$\ell_i^e = \Phi_i\,(H_i^e(\Psi_i\,(\vec{L}))) \tag{15}$$

or

$$\ell_i^e = \ell_i^e(H_i^e(L_i^e(\vec{L}))) \tag{16}$$

VI. Analytical Directional Growth Parameterization

Combining Eqs. (3), (10), (11), and (15) yields the following equation:

$$\rho_i = \rho(\vec{A}, \vec{L}) = \Psi_i(\vec{A}) * \Phi_i(H_i^e(\Psi_i(\vec{L}))) \tag{17}$$

Equation (17) gives an explicit relationship between the pole parameters and the element volume fraction. This equation is differentiable with respect to the pole parameters and is termed here the “Analytical Directional Growth Parameterization” (ADGP) equation.

A. Casting Designs

To generate designs that are suitable for casting, the ADGP equation can be used directly by picking the appropriate growth direction. In this case, the topology optimization problem is modified as follow:

$$\begin{aligned}
&\text{Min } F(\rho_1, \rho_2, \ldots, \rho_n) \\
&\text{such that:} \\
&g_j(\rho_1, \rho_2, \ldots, \rho_n) \le 0; \quad j = 1, m \\
&\rho_i = \rho_i(\vec{A}, \vec{L}) = \Psi_i(\vec{A}) * \Phi_i(H_i^e(\Psi_i(\vec{L}))); \quad i = 1, n \\
&0.0 \le A_k^p, L_k^p \le 1.0; \quad k = 1, NP
\end{aligned} \tag{18}$$

This problem is essentially the same as the one described in (1), as it considers the same objective function and the same constraints. However, the densities now are restricted to change with the pole parameters. The pole parameters become the independent design variables. When the optimization problem needs to evaluate the *F* or *g* functions, the ADGP equations are used to calculate the needed densities.

The initial values of the pole parameter should be 1.0 for L^p_k and the desired mass fraction for the A^p_k (values of 1.0 could be used instead but experience has shown the authors that choice wastes unnecessarily iterations).

It is interesting to mention here that this parametric topology optimization problem uses far fewer design variables than the original. The actual number of parameters depends on the pre-selected discretization parameter *h*. If *h* is picked to be the average size of an element, the reduction in the number of design variables is dramatically high. For example for a cube of 100*100*100 elements the number of pole design variables is 2*100*100, that is, 50 times smaller. This should not be surprising as the parameterization is essentially solving a 3D problem via a projected 2D problem.

B. Extrusion Designs

To obtain designs that represent extrusions, the ADGP equation is simplified by fixing the L_k^p variables to 1.0 or what is equivalent but more efficient is to only use the pole area densities parameters. In this case the reduced ADGP equation is:

$$\rho_i = \rho(\vec{A}) = \Psi_i(\vec{A}) \tag{19}$$

In this case the topology optimization problem is modified as follow:

$$\begin{aligned}
&\text{Min } F(\rho_1, \rho_2, \ldots, \rho_n) \\
&\text{such that:} \\
&g_j(\rho_1, \rho_2, \ldots, \rho_n) \le 0; \quad j = 1, m \\
&\rho_i = \rho_i(\vec{A}) = \Psi_i(\vec{A}); \quad i = 1, n \\
&0.0 \le A_k^p \le 1.0; \quad k = 1, NP
\end{aligned} \tag{20}$$

This reduced parametric topology optimization problem uses NP independent design variable, this is half as many design variables used by the parametric topology optimization problem proposed for casting designs.

VII. Practical Issues

The methodology described is this paper assumes that the material grows form the bottom of the design domain (lowest z coordinate). This automatically enforces that the parting plane be located there. This property of the method could be useful in some cases, but could lead to sub optimal designs. To use a parting plane at a fixed location other than the bottom of the design domain is simple: Use two design domains that touch each other on the desired location of the parting plane or surface, then select two growth directions that are parallel to each other but in opposite directions.

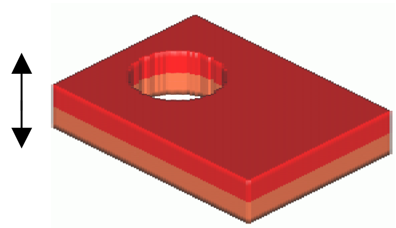


Figure 5. Design Space with two Designable Domains

In Fig. 5, the interface of the two domains can be used as a fixed parting-plane.

The presented methodology cannot find the optimal location of the parting plane. To achieve this, other equations should be used. The authors of this paper have developed such equations and plan to present them in a future paper.

VIII. Numerical Applications

A. Bar Subject to Torsional Loads

A simple 50 x 20 x 10 mm^3 flat bar shown in Fig 6 is used to illustrate the effectiveness of the ADGP method. The bar is fixed in one end and loaded with a twisting load in the other. The optimization problem consists of minimizing the global strain energy while keeping up to 30% of the material. The bar is modeled with a 20x8x4 mesh consisting of 640 hexahedral elements. Three cases are used to compare the design proposals that results from ignoring manufacturing requirements or imposing them.

In the first case, a standard topology optimization was performed using 640 independent design variables.

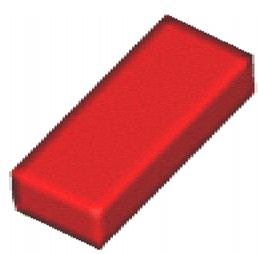
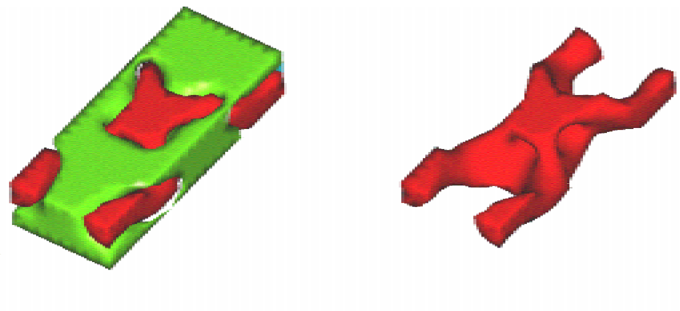


Figure 6. Design Space

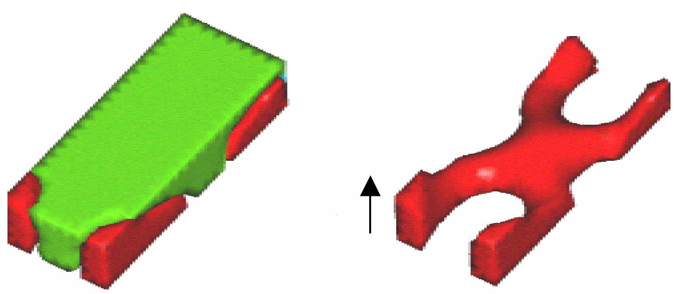
Figure 7a shows the topology results. In this figure, the red colored (dark) part corresponds to the solid material, which is the material to keep, while the green (light) part corresponds to the void material, which is the material to discard. Figure 7b shows only the solid material. In Fig. 7b, two holes are revealed, the first is in the direction of the main axis of the beam and the second is perpendicular to it. The first hole tries to make the structure a tube, but since there is not enough material for that, the second hole carves out more material. Because of these two holes, this design proposal cannot be manufactured with a simple casting process, which is an example of why manufacturing requirements may be needed.



a) Solid and Void Material b) Solid Material

Figure 7. Standard Topology Results

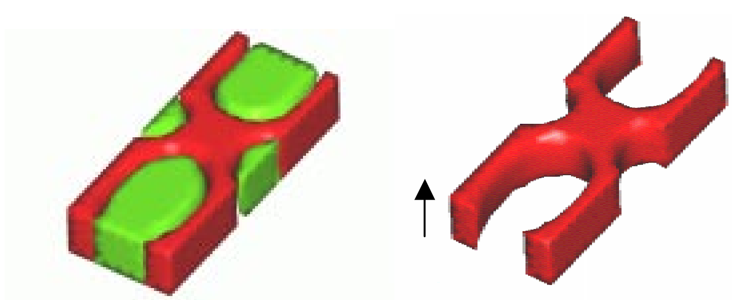
In the second case the ADGP method was used to find a design proposal suitable for casting. In total 320 (160*2) independent design variables were automatically created. Figure 8a shows the topology results, in red (dark) is the solid material and in green (light) is the void material. The void material in Fig. 8a reveals the form of a potential mold. Figure 8b shows the solid material corresponding to the topology final design. The final results show that the structure does not have interior holes so this structure can be manufactured using a casting process.



a) Solid and Void Material b) Solid Material

Figure 8. Topology Results for Casting

In the third case the reduced ADGP method was used to design the part assuming extrusion requirements. In this case 160 independent design variables were used. Figure 9a shows the final topology results, in red are the solid material; in green is the void material. Figure 9b shows the solid material corresponding to the topology final design. The topology optimization results produce no interior holes, and the material either completely fills the design domain in the z direction or not at all. This structure can be produced using an extrusion process, as desired.



a) Solid and Void Material b) Solid Material

Figure 9. Topology Results for Extrusion

B. Michell-Truss Like Problem

The classical 2D Michell truss problem[19] is commonly used to verify topology optimization algorithms[17, 20]. The problem consists in finding the optimal location of material to support a vertical load applied on one end of the plate. The plate is supported by fixing points on a circular pattern next to the other end. In the three dimensional version of the original Michell problem the vertical load is replaced by a twisting load and the design domain is a large sphere. In this paper we use a variation of the 2D type, which consists of considering a thickness in the domain and using a 3D representation of this domain to allow for a thickness variation. The exterior dimensions of the block are 55x40x10, the radius of the circular hole is 10. The finite element mesh used in this problem contains 60,704 hexahedral elements, 70,722 grids and 208,251 degrees of freedoms. The optimization problem consists of minimizing the global strain energy while keeping 10% of the material.

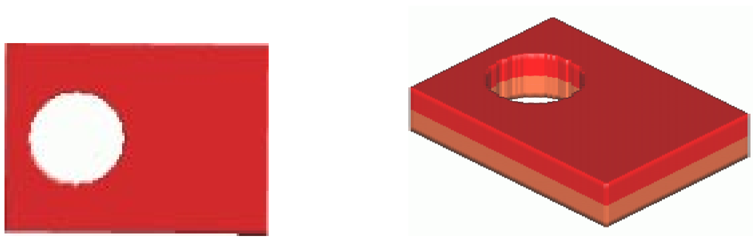


Figure 10. Design Space

In the first case, a standard topology optimization was performed using 60,704 independent design variables, i.e. one design variable per designable element. The optimization process took 22 design cycles to converge. Figure 11 shows different views of the topology results. The obtained design proposal looks reasonable as the top view of the structure looks similar to the classical planar solution. However, the solution is hollow so it cannot be manufactured with a simple casting.

Figure 11. Standard Topology Results

In the second case, two design domains and two sets of manufacturing requirements using the ADGP method are used to find a design proposal suitable for casting. The first design domain was used to grow the structure from a middle plane up, while the other was used to grow the structure from the middle plane down. The two arrows pointing in opposite directions in Fig. 12 show the two growth directions.

In total, 7936 (1984*2*2) independent design variables were used. In other words, the number of variable used is 87% less than the number of variables used on the first case. In this case the optimization process took 25 design cycles to converge. Figure 12 shows the topology results. The final results show that the structure does not have interior holes, as desired; so it can be manufactured using a casting process.

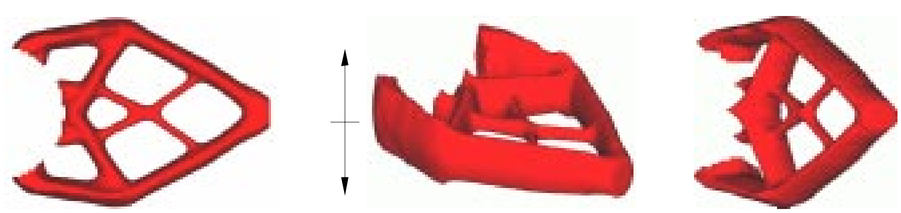


Figure 12. Topology Results for Casting

IX. Conclusions

A method to produce castable or extrudable topology design proposals has been presented. The method allows for controlling minimum sizes that are also important for manufacturing. Numerically, the method avoids mesh dependencies and controls the checkerboard phenomena. The method is suitable to be used with optimization programs based on mathematical nonlinear programming optimization. The method presented here is more efficient than alternative methods since it uses fewer design variables than others and does not need to deal directly with mathematical constraints that simulate the manufacturing requirements. The proposed approach was implemented in the GENESIS program.

Although the presented method was originally developed to be use with 3D structures, it can be used in 2D structures as well. If the growth direction is contained in the plane of the structure then the method can be used to design planar castable structures. If the growth direction is normal to the plane of the 2D structure then the method can be used to impose minimum member size and to avoid checkerboard problems.

In the presented method the material growth is considered from the bottom of the design domain. For certain problems this can be a limitation and may produce suboptimal solutions, as the location of the parting plane might be unknown. The authors of this paper have developed a set of equations that overcome this limitation and expect to present it in a future publication.

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